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# Thiele-type branched continued fractions for two-variable functions

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## 1. Introduction

We often expect that interpolation and approximation by rational functions yield better results than interpolation and approximation by polynomials. It is obvious in problems where the interpolated function is a rational function and it is natural in problems for which such a function is meromorphic.

The same is true if we deal with multivariable functions. In this case however, the situation is much more complex. For example, Padé approximants which are closely connected with both interpolation and approximation problems, can not be simply generalized for several variable functions. Therefore, there are many definitions of Padé-type multivariable approximants (cf. [1–6]). Roughly speaking, each of them has been formulated by modifying one of the conditions fulfilled by one-variable Padé approximants.

On the other hand, we know how closely continued fractions are connected with interpolation and approximation problems, for example, via Padé approximants. Therefore, it seems to be natural that generalizations of continued fractions for multivariable functions (so-called branched continued fractions) will play a similar role in multivariable problems as ordinary continued fractions play in one-variable theory.

The most general form of branched continued fractions (henceforth BCF) is given in W.Ja. Skorobogatko's book [7]. It is a BCF of the form

$$a_0+\sum_{k_0=1}^{N}\cfrac{a_{k_0}^{(1)}}{a_{k_0,0}^{(1)}+\sum_{k_1=1}^{N}\cfrac{a_{k_0k_1}^{(2)}}{a_{k_0k_1,0}^{(2)}+\sum_{k_2=1}^{N}\cfrac{a_{k_0k_1k_2}^{(3)}}{a_{k_0k_1k_2,0}^{(3)}+\ddots+\sum_{k_{m-1}=1}^{N}\cfrac{a_{k_0k_1\cdots k_{m-1}}^{(m)}}{a_{k_0k_1\cdots k_{m-1},0}^{(m)}+\ddots}}}} \tag{1.1}$$

where $N$ corresponds to the number of variables.

This general form is not very convenient for practical use. Moreover we have some troubles with the uniqueness of a BCF of the form (1.1) corresponding to a given $N$-variable function.

More useful and simpler are two particular cases of BCF (1.1). One of them is given independently by O'Donohoe [9] and Kutschminskaja [8]. It is a BCF of the following form (here we give a function form of this BCF for a two-dimensional case):

$$b_0+K_1^0(x)+K_2^0(y)+\frac{a_1xy}{b_1+K_1^1(x)+K_2^1(y)}+\frac{a_2xy}{b_2+K_1^2(x)+K_2^2(y)}+\cdots \tag{1.2}$$

where $K_j^i(t_j)$, $i = 0, 1, \ldots$, $j = 1, 2$, $t_1 = x$, $t_2 = y$, are continued fractions

$$K_j^i(t_j) = b_{j,0}^i + \frac{t_j}{b_{j,1}^i} \, \frac{t_j}{+\; b_{j,2}^i} \, + \cdots . \tag{1.3}$$

The other BCF is given in [10]. It is a BCF of the form (with the same restrictions as above):

$$K_0(xy) + \left[ \frac{a_1^1 x}{K_1^1(xy)} \, \frac{a_2^1 x}{+\; K_2^1(xy)} \, + \cdots \right] + \left[ \frac{a_1^2 y}{K_1^2(xy)} \, \frac{a_2^2 y}{+\; K_2^2(xy)} \, + \cdots \right] \tag{1.4}$$

where for $i = 1, 2$, $j = 0, 1, \ldots$

$$K_j^i(xy) = b_{j,0}^i + \frac{xy}{b_{j,1}^i} \, \frac{xy}{+\; b_{j,2}^i} \, + \cdots . \tag{1.5}$$

These two forms of BCFs are obtained by suitable decompositions of the given two-variable power series if we expand the series into a corresponding BCF. The connection between structures of two-variable power series and the above-presented BCFs will be illustrated in the following way: if we imagine the components of a two-variable power series as nodes of an infinite grid as shown in Fig. 1 (we mean that each node is occupied by a respective $c_{kl}x^k y^l$), then $b_p + K_1^p(x) + K_2^p(y)$, $p = 0, 1, \ldots$ in (1.2) corresponds

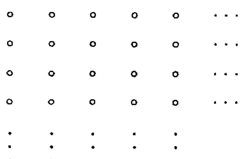

Fig. 1.

to the $p$th 'path' shown in Fig. 2, while $K_j^i(xy)$, $i = 1, 2, j = 0, 1, \ldots$ corresponds to the $(i, j)$th diagonal, as shown in Fig. 3.

For the three- or more-variable case the two above-presented BCFs become much more complicated with respect to the necessary decompositions of a multivariable power series into respective components. To get rid of these problems we will make one more simplification of the BCF (1.1) by taking into account

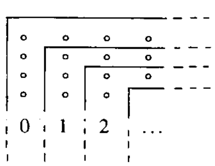


Fig. 2.

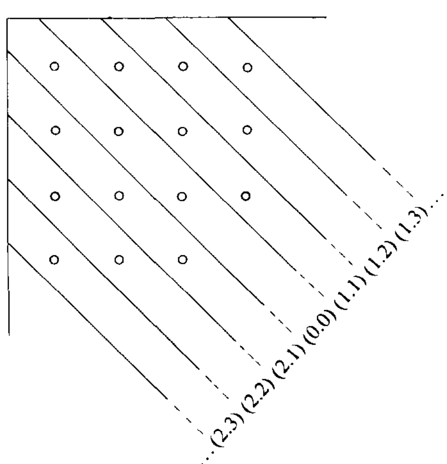


Fig. 3.

the scheme for obtaining Hartogs power series from a given multivariable power series. More precisely, if

$$f(z)=\sum_{|\nu|=0}^{\infty} c_\nu z^\nu \tag{1.6}$$

is the given $N$-variable power series, where $z=(z_1,\dots,z_N)\in\mathbb{C}^N$, $\nu=(\nu_1,\dots,\nu_N)\in\mathbb{Z}_+^N$, $|\nu|=\sum_{j=1}^N \nu_j$, $z^\nu=z_1^{\nu_1}\cdots z_N^{\nu_N}$, then we will find a continued fraction of the form

$$K_0('z)+\frac{a_1 z_1}{K_1('z)}\;\frac{}{+}\;\frac{a_2 z_1}{K_2('z)}\;\frac{}{+}\;\cdots \tag{1.7}$$

for which $'z=(z_2,\dots,z_N)\in\mathbb{C}^{N-1}$ and each $K_i('z)$ are $(N-1)$-variable power series, and which corresponds to the power series (1.6). Obviously all $K_i('z)$ can be expanded into respective continued fractions of the form

$$K_0^i(''z)+\frac{a_1^i z_2}{K_1^i(''z)}\;\frac{}{+}\;\frac{a_2^i z_2}{K_2^i(''z)}\;\frac{}{+}\;\cdots \tag{1.8}$$

with new $(N-2)$-variable power series $K_j^i(''z)$, $''z=(z_3,\dots,z_N)$, and so on.

For the two-dimensional case this gives BCFs of the form

$$K_0(y)+\frac{a_1 x}{K_1(y)}\;\frac{}{+}\;\frac{a_2 x}{K_2(y)}\;\frac{}{+}\;\cdots \tag{1.9}$$

where for $j=0, 1,\dots$

$$K_j(y)=b_0^j+\frac{a_1^j y}{b_1^j}\;\frac{}{+}\;\frac{a_2^j y}{b_2^j}\;\frac{}{+}\;\cdots. \tag{1.10}$$

Bearing in mind the graphical illustrations for BCFs (1.2) and (1.4) we may say that in (1.9) $K_j(y)$ corresponds to the $j$th vertical column in Fig. 4.

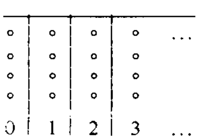

Fig. 4.

In the following sections we will show how BCFs obtained in such a way can be used for two-variable interpolation and approximation problems. We will use the following presentation for the BCF type (1.9):

$$\left[b_0^0+\frac{a_1^0}{b_1^0}\;\frac{}{+}\;\frac{a_2^0}{b_2^0}\;\frac{}{+}\;\cdots\right]+\frac{a_1}{\left[b_0^1+\frac{a_1^1}{b_1^1}\;\frac{}{+}\;\frac{a_2^1}{b_2^1}\;\frac{}{+}\;\cdots\right]}\;\frac{}{+}\;\frac{a_2}{\left[b_0^2+\frac{a_1^2}{b_1^2}\;\frac{}{+}\;\frac{a_2^2}{b_2^2}\;\frac{}{+}\;\cdots\right]}\;\frac{}{+}\;\cdots. \tag{1.11}$$

The expression

$$T_{n,m}=\frac{P_{n,m}}{Q_{n,m}}=\left[b_0^0+\frac{a_1^0}{b_1^0}\;\frac{}{+}\;\cdots\;\frac{}{+}\;\frac{a_m^0}{b_m^0}\right]+\frac{a_1}{\left[b_0^1+\frac{a_1^1}{b_1^1}\;\frac{}{+}\;\cdots\;\frac{}{+}\;\frac{a_m^1}{b_m^1}\right]}$$

$$+\cdots+\frac{a_n}{\left[b_0^n+\frac{a_1^n}{b_1^n}\;\frac{}{+}\;\cdots\;\frac{}{+}\;\frac{a_m^n}{b_m^n}\right]} \tag{1.12}$$

is called the $(n, m)$th approximant of BCF (1.11).

## 2. Two-variable rational interpolation

Let $f(x, y)$ be a function defined on the rectangle $\Delta = (a, b) \times (c, d) \subset \mathbb{R}^2$ and $\Pi_x^n = \{x_i, i = 0, 1, \ldots, n\}$ a given partition of the interval $(a, b)$. If we consider the restricted function $f(\cdot, \bar{y})$, where $\bar{y} \in (c, d)$ is arbitrarily chosen, we will search for a rational function $r_n(x, \bar{y}) = p_n(x, \bar{y})/q_n(x, \bar{y})$, depending on the variable $x$, such that $\deg p_n(x, \bar{y}) = [\frac{1}{2}(n+1)]$, $\deg q_n(x, \bar{y}) = [\frac{1}{2}n]$ and $r_n(x, \bar{y})$ interpolates the one-variable function $f(\cdot, \bar{y})$ at points $x_i \in \Pi_x^n$,

$$r_n(x_i, \bar{y}) = f(x_i, \bar{y}), \quad i = 0, 1, \ldots, n. \tag{2.1}$$

Let us assume that such a rational function exists. We know that it will be found as $n$th approximant of a corresponding Thiele's continued fraction [11] of the form

$$r_n(x, \bar{y}) = b_0(\bar{y}) + \frac{x - x_0}{b_1(\bar{y})} + \cdots + \frac{x - x_{n-1}}{b_n(\bar{y})}. \tag{2.2}$$

Obviously the coefficients $b_i(\bar{y})$ depend on the variable $y$.

Further, let $\Pi_y^m = \{y_j, j = 0, 1, \ldots, m\}$ be a given partition of the interval $(c, d)$. We will form a rectangular grid $\Pi_{x,y}^{n,m} = \Pi_x^n \times \Pi_y^m$ and now we will seek for a two-variable rational function $R_{n,m}(x, y)$ depending on $(n+1)(m+1)$ coefficients, such that

$$R_{n,m}(x_i, y_j) = f_{ij} = f(x_i, y_j), \quad i = 0, 1, \ldots, n, j = 0, 1, \ldots, m, \tag{2.3}$$

and such that for each $y_j$, $0 \leqslant j \leqslant m$, $R_{n,m}(x, y_j)$ is the solution of the rational interpolation problem with nodes $x_i$, $i = 0, 1, \ldots, n$, and respective node values equal $f_{ij}$, $i = 0, 1, \ldots, n$.

If such rational function exists, then it will have the form

$$R_{n,m}(x, y) = b_0(y) + \frac{x - x_0}{b_1(y)} + \cdots + \frac{x - x_{n-1}}{b_n(y)} \tag{2.4}$$

where all $b_j(y), j = 0, 1, \ldots, n$, are polynomials or rational functions of the variable $y$ and the total number of their coefficients equals $(n+1)(m+1)$. Therefore we will formulate the following interpolation problem:

Let $\Pi_x^n \times \Pi_y^m$ be a rectangular grid contained in the rectangle $\Delta$ and let $f(x, y)$ be a function defined on $\Delta$ such that

$$f(x_i, y_j) = f_{ij}, \quad i = 0, 1, \ldots, n, j = 0, 1, \ldots, m. \tag{2.5}$$

We are looking for a rational function $R_{n,m}(x, y) = P_{n,m}(x, y)/Q_{n,m}(x, y)$ which has the form (2.4) where $b_j(y) = p_j(y)/q_j(y), j = 0, 1, \ldots, n$, are rational functions for which

$$\deg p_j(y) \leqslant k < m, \qquad \deg q_j(y) \leqslant m - k, \quad j = 0, 1, \ldots, n, \tag{2.6}$$

and, moreover, such that

$$R_{n,m}(x_i, y_j) = f_{ij}, \quad i = 0, 1, \ldots, n, j = 0, 1, \ldots, m. \tag{2.7}$$

Simple examples show that there doesn't exist a rational function for which equations (2.7) are fulfilled. Therefore, we will replace (2.7) by equations

$$f_{ij} Q_{n,m}(x_i, y_j) - P_{n,m}(x_i, y_j) = 0, \quad i = 0, 1, \ldots, n, j = 0, 1, \ldots, m, \tag{2.8}$$

as it is usually done for similar one-variable interpolation problems. Now we will prove the existence and uniqueness of the rational function $R_{n,m}(x, y)$ of the form (2.4) for which conditions (2.6) and (2.8) are fulfilled.

So, if $k$ in (2.6) is given then we can easily see that there exists a unique rational function $b_0(y) = p_0(y)/q_0(y)$ with degrees of the numerator and the denominator as in (2.6) and such that

$$f_{0,j} q_0(y_j) - p_0(y_j) = 0, \quad j = 0, 1, \ldots, m. \tag{2.9}$$

Further, let us assume that there exist rational functions $b_i(y)=p_i(y)/q_i(y)$, $i=0, 1,\dots, l-1<n$, such that for a two-variable function

$$R_{l-1,m}(x,y)=\frac{P_{l-1,m}(x,y)}{Q_{l-1,m}(x,y)}$$
$$=b_0(y)+\frac{x-x_0}{b_1(y)}+\dots+\frac{x-x_{l-2}}{b_{l-1}(y)} \tag{2.10}$$

we have

$$f_{ij}Q_{l-1,m}(x_i,y_j)-P_{l-1,m}(x_i,y_j)=0,\quad i=0, 1,\dots, l-1, j=0, 1,\dots, m. \tag{2.11}$$

Now we will form a system of $(m+1)$ homogeneous linear equations with $(m+2)$ unknown coefficients of the rational function $b_l(y)=p_l(y)/q_l(y)$

$$q_1(y_j)(x_l-x_{l-1})\left[f(x_l,y_j)Q_{l-2}(x_l,y_j)-P_{l-2}(x_l,y_j)\right]$$
$$+p_l(y_j)\left[f(x_l,y_j)Q_{l-1}(x_l,y_j)-P_{l-1}(x_l,y_j)\right]=0,\quad j=0, 1,\dots, m. \tag{2.12}$$

This system has a nontrivial solution and the rational function $b_l(y)=p_l(y)/q_l(y)$ is uniquely defined. On the other hand, if we put $x_i$, $i=0, 1,\dots, l-1$ instead of $x_l$ into (2.12) then equations (2.12) remain true for $j=0, 1,\dots, m$.

We will introduce the following notations:

$$P_l(x,y)=p_l(y)P_{l-1}(x,y)+(x-x_{l-1})P_{l-2}(x,y)q_l(y),$$
$$Q_l(x,y)=p_l(y)Q_{l-1}(x,y)+(x-x_{l-1})Q_{l-2}(x,y)q_l(y). \tag{2.13}$$

Now from (2.12)

$$f_{ij}Q_l(x_i,y_j)-P_l(x_i,y_j)=0,\quad i=0, 1,\dots, l, j=0, 1,\dots, m, \tag{2.14}$$

and from the general theory of continued fractions, $P_l(x, y)$ and $Q_l(x, y)$ can be treated respectively as the numerator and the denominator of a continued fraction

$$\frac{P_l(x,y)}{Q_l(x,y)}=b_0(y)+\frac{x-x_0}{b_1(y)}+\dots+\frac{x-x_{l-1}}{b_l(y)}. \tag{2.15}$$

Since all rational functions $b_i(y)$ exist and are unique it leads to the following theorem.

**Theorem 1.** *For every rectangular grid $\Pi^{n,m}_{x,y}$ there exist uniquely defined rational functions $b_i(y)=p_i(y)/q_i(y)$, $i=0, 1,\dots, n$, such that*

$$\deg p_i(y)\leqslant k<m,\qquad \deg q_i(y)\leqslant m-k, \tag{2.16}$$

*and such that for the two-variable rational function of the form*

$$R_{n,m}(x,y)=\frac{P_{n,m}(x,y)}{Q_{n,m}(x,y)}=b_0(y)+\frac{x-x_0}{b_1(y)}+\dots+\frac{x-x_{n-1}}{b_n(y)} \tag{2.17}$$

*the following equations are fulfilled:*

$$f_{ij}Q_{n,m}(x_i,y_j)-P_{n,m}(x_i,y_j)=0,\quad i=0, 1,\dots, n, j=0, 1,\dots, m. \tag{2.18}$$

In the following section we will examine the case when

$$\deg p_i(y)\leqslant\left[\frac{m+1}{2}\right],\qquad \deg q_i(y)\leqslant\left[\frac{m}{2}\right],\quad i=0, 1,\dots, m. \tag{2.19}$$

## 3. Partial inverted differences

Let $\Pi_{x,y}^{n,m} \subset \Delta \subset \mathbb{R}^2$ be a given rectangular grid contained in the rectangle $\Delta$, and let $f(x, y)$ be a real function defined on $\Delta$ and such that

$$f(x_i, y_j) = f_{ij}, \quad i = 0, 1, \ldots, n, j = 0, 1, \ldots, m. \tag{3.1}$$

We will introduce the following notations:

$$\varphi_{0,0}(x_i; y_k) = f_{ik}, \qquad \varphi_{1,0}(x_i, x_j; y_k) = \frac{x_i - x_j}{\varphi_{0,0}(x_i; y_k) - \varphi_{0,0}(x_j; y_k)}$$

and for $s \geq 1$

$$\varphi_{s+1,0}(x_{p_1}, \ldots, x_{p_s}, x_i, x_j; y_k) = \frac{x_i - x_j}{\varphi_{s,0}(x_{p_1}, \ldots, x_{p_s}, x_i; y_k) - \varphi_{s,0}(x_{p_1}, \ldots, x_{p_s}, x_j; y_k)}.$$

Further, let

$$\varphi_{s,1}(x_{p_0}, x_{p_1}, \ldots, x_{p_s}; y_k, y_l) = \frac{y_k - y_l}{\varphi_{s,0}(x_{p_0}, \ldots, x_{p_s}; y_k) - \varphi_{s,0}(x_{p_0}, \ldots, x_{p_s}; y_l)}$$

and for $r \geq 1$

$$\varphi_{s,r+1}(x_{p_0}, \ldots, x_{p_s}; y_{q_1}, \ldots, y_{q_r}, y_k, y_1) = \frac{y_k - y_l}{\varphi_{s,r}(x_{p_0}, \ldots, x_{p_s}; y_{q_1}, \ldots, y_{q_r}, y_k) - \varphi_{s,r}(x_{p_0}, \ldots, x_{p_s}; y_{q_1}, \ldots, y_{q_r}, y_l)}.$$

If for $\{x_{p_0}, x_{p_1}, \ldots, x_{p_i}\} \subset \Pi_x^n$ and $\{y_{q_0}, y_{q_1}, \ldots, y_{q_j}\} \subset \Pi_j^m$ there exists $\varphi_{i,j}(x_{p_0}, \ldots, x_{p_i}; y_{q_0}, \ldots, y_{q_j})$, then it is called the $(i, j)$th partial inverted difference for function $f(x, y)$.

Together with partial inverted differences we will consider their symmetrizations:

$$\rho_{0,0}(x_i; y_j) = f_{ij}, \qquad \rho_{1,0}(x_k, x_l; y_j) = \frac{x_k - x_l}{\rho_{0,0}(x_k; y_j) - \rho_{0,0}(x_l; y_j)},$$

and for $p \geq 2$

$$\rho_{p,0}(x_k, x_{k+1}, \ldots, x_{k+p}; y_j) = \rho_{p-2,0}(x_{k+1}, \ldots, x_{k+p-1}; y_j) + \frac{x_k - x_{k+p}}{\rho_{p-1,0}(x_k, \ldots, x_{k+p-1}; y_j) - \rho_{p-1,0}(x_{k+1}, \ldots, x_{k+p}; y_j)}.$$

Similarly

$$\rho_{p,1}(x_k, x_{k+1}, \ldots, x_{k+p}; y_r, y_s) = \frac{y_r - y_s}{\rho_{p,0}(x_k, \ldots, x_{k+p}; y_r) - \rho_{p,0}(x_k, \ldots, x_{k+p}; y_s)},$$

and for $q \geq 2$

$$\rho_{p,q}(x_k, \ldots, x_{k+p}; y_l, \ldots, y_{l+q}) = \rho_{p,q-2}(x_k, \ldots, x_{k+p}; y_{l+1}, \ldots, y_{l+q-1}) + \frac{y_l - y_{l+q}}{\rho_{p,q-1}(x_k, \ldots, x_{k+p}; y_l, \ldots, y_{l+q-1}) - \rho_{p,q-1}(x_k, \ldots, x_{k+p}, y_{l+1}, \ldots, y_{l+q})}.$$

If $\rho_{p,q}(x_k, \ldots, x_{k+p};\ y_l, \ldots, y_{l+q})$ exists then it is called the $(p, q)$th partial reciprocal difference for function $f(x, y)$.

It is easy to prove two following lemmas.

**Lemma 1.** *If respective partial reciprocal differences exist then for* $0 \leq \alpha < \beta \leq n$, $m \geq 0$, *we have*

$$\rho_{n,m}(x_0, \ldots, x_\alpha, \ldots, x_\beta, \ldots, x_n; y_0, y_1, \ldots, y_m) = \rho_{n,m}(x_0, \ldots, x_\beta, \ldots, x_\alpha, \ldots, x_n; y_0, y_1, \ldots, y_m) \tag{3.2}$$

*and similarly for* $0 \leqslant \alpha < \beta \leqslant m$, $n \geqslant 0$

$$\rho_{n,m}(x_0, x_1, \ldots, x_n; y_0, \ldots, y_\alpha, \ldots, y_\beta, \ldots, y_m) = \rho_{n,m}(x_0, x_1, \ldots, x_n; y_0, \ldots, y_\beta, \ldots, y_\alpha, \ldots, y_m). \tag{3.3}$$

**Lemma 2.** *If the respective partial reciprocal differences and partial inverted differences exist, then for* $k \geqslant 1$

$$\varphi_{k,0}(x_0, \ldots, x_k; y_r) = \rho_{k,0}(x_0, \ldots, x_k; y_r) - \rho_{k-2,0}(x_0, \ldots, x_{k-2}; y_r) \tag{3.4}$$

*and for* $l \geqslant 1$

$$\varphi_{k,l}(x_0, \ldots, x_k; y_0, \ldots, y_l) = \rho_{k,l}(x_0, \ldots, x_k; y_0, \ldots, y_l) - \rho_{k,l-2}(x_0, \ldots, x_k; y_0, \ldots, y_{l-2}) \tag{3.5}$$

*where* $\rho_{-1,0} = \rho_{k,-1} = 0$.

We delete proofs of the above lemmas since they are easy but long modifications of the corresponding proofs in the one-dimensional case.

With the use of the above defined partial inverted differences we will find the decomposition of the given function $f(x, y)$ defined on $\Delta$ into branched continued fraction.

**Lemma 3** (Thiele-type interpolation formula). *If all the partial inverted differences* $\varphi_{k,l}(x_0, \ldots, x_k; y_0, \ldots, y_l)$, $0 \leqslant k \leqslant n$, $0 \leqslant l \leqslant m$, *for the function* $f(x, y)$ *are defined then*

$$\begin{aligned}
f(x,y) = {} & \left[\varphi_{0,0}(x_0; y_0) + \frac{y-y_0}{\varphi_{0,1}(x_0; y_0, y_1)} + \cdots + \frac{y-y_{m-1}}{\varphi_{0,m}(x_0; y_0, \ldots, y_m)} + \frac{y-y_m}{\varphi_{0,m+1}(x_0; y_0, \ldots, y_m, y)}\right] \\
& + \frac{x-x_0}{\left[\varphi_{1,0}(x_0, x_1; y_0) + \frac{y-y_0}{\varphi_{1,1}(x_0, x_1; y_0, y_1)} + \cdots + \frac{y-y_{m-1}}{\varphi_{1,m}(x_0, x_1; y_0, \ldots, y_m)} + \frac{y-y_m}{\varphi_{1,m+1}(x_0, x_1; y_0, \ldots, y_m, y)}\right]} \\
& \dotplus \frac{x-x_{n-1}}{\left[\varphi_{n,0}(x_0, \ldots, x_n; y_0) + \frac{y-y_0}{\varphi_{n,1}(x_0, \ldots, x_n; y_0, y_1)} + \cdots\right.} \\
& \qquad \left.\cdots + \frac{y-y_{m-1}}{\varphi_{n,m}(x_0, \ldots, x_n; y_0, \ldots, y_m)} + \frac{y-y_m}{\varphi_{n,m+1}(x_0, \ldots, x_n; y_0, \ldots, y_m, y)}\right] \\
& + \frac{x-x_n}{\left[\varphi_{n+1,0}(x_0, \ldots, x_n, x; y_0) + \frac{y-y_0}{\varphi_{n+1,1}(x_0, \ldots, x_n, x; y_0, y_1)} + \cdots\right.} \\
& \qquad \left.\cdots + \frac{y-y_{m-1}}{\varphi_{n+1,m}(x_0, \ldots, x_n, x; y_0, \ldots, y_m)} + \frac{y-y_m}{\varphi_{n+1,m+1}(x_0, \ldots, x_n, x; y_0, \ldots, y_m, y)}\right].
\end{aligned} \tag{3.6}$$

*Moreover, if* $T_{k,l}(x, y)$ *denotes the* $(k, l)$*th approximant of BCF* (3.6), $0 \leqslant k \leqslant n$, $0 \leqslant l \leqslant m$, *then*

$$T_{k,l}(x_i, y_j) = f_{ij}, \quad i = 0, 1, \ldots, k, j = 0, 1, \ldots, l. \tag{3.7}$$

**Proof.** If we reduce continued fractions in (3.6) according to the definition of inverted differences then we obtain the first equality. For the proof of (3.7) let us notice that

$$\varphi_{i,0}(x_0, \ldots, x_i; y_0) + \frac{y_j - y_0}{\varphi_{i,1}(x_0, \ldots, x_i; y_0, y_1)} + \cdots + \frac{y_j - y_{l-1}}{\varphi_{i,l}(x_0, \ldots, x_i, y_0, \ldots, y_l)}$$

$$= \varphi_{i,0}(x_0,\dots,x_i;y_0) + \frac{y_j - y_0}{\varphi_{i,1}(x_0,\dots,x_i;y_0,y_1)} + \cdots + \frac{y_j - y_{j-1}}{\varphi_{i,j}(x_0,\dots,x_i;y_0,\dots,y_j)}$$

$$= \varphi_{i,j}(x_0,\dots,x_i;y_j).$$

Thus, we have

$$T_{k,l}(x_i,y_j) = \varphi_{0,j}(x_0,y_j) + \frac{x_i - x_0}{\varphi_{1,j}(x_0,x_1;y_j)} + \cdots + \frac{x_i - x_{i-1}}{\varphi_{i,j}(x_0,\dots,x_i;y_j)} = f_{ij},$$

for $i = 0, 1,\dots,k$, $j = 0, 1,\dots,l$. □

Equation (3.6) together with (3.7) is a two-variable interpolation formula for interpolating the given function $f(x, y)$ on the rectangular grid $\Pi^{n,m}_{x,y}$ by a two-variable rational function of our special form. Formula (3.6) is called a Thiele-type two-variable interpolation formula for the function $f(x, y)$ on the rectangular grid $\Pi^{n,m}_{x,y}$.

## 4. Error formula for interpolation

In this section let us assume $\Pi^n_x = \{x_1,\dots,x_n\}$, $\Pi^m_y = \{y_1,\dots,y_m\}$ (we omit points $x_0$ and $y_0$). As in previous sections we will form a rectangular grid $\Pi^n_x \times \Pi^m_y$.

If $f(x, y)$ is a function defined on $\Delta$, such that

$$f(x_i,y_j) = f_{ij}, \quad i = 1,\dots,n, j = 1,\dots,m, \tag{4.1}$$

and all partial inverted differences $\varphi_{k,l}(x_1,\dots,x_{k+1};\ y_1,\dots,y_{l+1})$, $0 \leqslant k \leqslant n$, $0 \leqslant l \leqslant m$, for $f(x, y)$ are defined on $\Pi^n_x \times \Pi^m_y$ then from Lemma 3 the rational function $T_{n-1,m-1}(x, y)$ defined as

$$T_{n-1,m-1}(x,y) = \frac{P_{n-1,m-1}(x,y)}{Q_{n-1,m-1}(x,y)} = L_0(y) + \frac{x - x_1}{L_1(y)} + \cdots + \frac{x - x_{n-1}}{L_{n-1}(y)} \tag{4.2}$$

where

$$L_i(y) = \left[\varphi_{i,0}(x_1,\dots,x_{i+1};y_1) + \frac{y - y_1}{\varphi_{i,1}(x_1,\dots,x_{i+1};y_1,y_2)} + \cdots \right.$$
$$\left. + \frac{y - y_{m-1}}{\varphi_{i,m-1}(x_1,\dots,x_{i+1};y_1,\dots,y_m)}\right],$$

interpolates $f(x, y)$ in corresponding nodes $(x_i, y_j)$, $i = 1,\dots,n$, $j = 1,\dots,m$. Together with $T_{n-1,m-1}(x, y)$ we will examine the function

$$T_{n-1,m}(x,y) = \frac{P_{n-1,m}(x,y)}{Q_{n-1,m}(x,y)} = K_0(y) + \frac{x - x_1}{K_1(y)} + \cdots + \frac{x - x_{n-1}}{K_{n-1}(y)} \tag{4.3}$$

where

$$K_i(y) = \left[\varphi_{i,0}(x_1,\dots,x_{i+1};y_1) + \frac{y - y_1}{\varphi_{i,1}(x_1,\dots,x_{i+1};y_1,y_2)} + \cdots \right.$$
$$\left. + \frac{y - y_{m-1}}{\varphi_{i,m-1}(x_1,\dots,x_{i+1};y_1,\dots,y_m)} + \frac{y - y_m}{\varphi_{i,m}(x_1,\dots,x_{i+1};y_1,\dots,y_m,y)}\right].$$

It is easy to see that $T_{n-1,m}(x, y)$ is the $(n-1)$th approximant of the Thiele type continued fraction corresponding to the restricted interpolation problem consisting in finding a rational function interpolating the one-variable function $f(\cdot, y)$ at points $x_i$, $i = 1,\dots,n$.

If we define the function

$$F(u,y)=Q_{n-1,m}(x,y)[f(x,y)-T_{n-1,m}(u,y)]$$
$$-Q_{n-1,m}(x,y)[f(x,y)-T_{n-1,m}(x,y)]\frac{\omega_n^1(u)}{\omega_n^1(x)} \tag{4.4}$$

where $\omega_n^1(x)=\prod_{j=1}^{n}(x-x_j)$, then it vanishes at $(n+1)$ points $x_1,\dots,x_n$, $x$. Therefore, if $f(x,y)\in\mathcal{C}_\Delta^{n+m}$, then from the Rolle theorem there exists a point $\xi_0\in(a,b)$ such that

$$f(x,y)-T_{n-1,m}(x,y)=\frac{\omega_n^1(x)}{n!Q_{n-1,m}(x,y)}\mathrm{D}_x^n[Q_{n-1,m}(x,y)f(x,y)]_{x=\xi_0} \tag{4.5}$$

where $\mathrm{D}_x^n$ is the symbol of differentiation. Further, since

$$T_{n-1,m-1}(x_i,y_j)=f_{ij},\quad i=1,\dots,n, j=1,\dots,m, \tag{4.6}$$

we have

$$T_{n-1,m}(x,y_j)-T_{n-1,m-1}(x,y_j)=0,\quad j=1,\dots,m, \tag{4.7}$$

for an arbitrarily chosen $x\in(a,b)$. Therefore, there exists $\eta_0\in(c,d)$ such that

$$\begin{aligned}
&T_{n-1,m}(x,y)-T_{n-1,m-1}(x,y)=\\
&=\frac{\omega_m^2(y)}{m!Q_{n-1,m}(x,y)Q_{n-1,m-1}(x,y)}\\
&\quad\times\mathrm{D}_y^m\{Q_{n-1,m}(x,y)Q_{n-1,m-1}(x,y)[T_{n-1,m}(x,y)-T_{n-1,m-1}(x,y)]\}_{y=\eta_0}\\
&=\frac{\omega_m^2(y)}{m!Q_{n-1,m}(x,y)Q_{n-1,m-1}(x,y)}\\
&\quad\times\{\mathrm{D}_y^m[Q_{n-1,m}(x,y)Q_{n-1,m-1}(x,y)(T_{n-1,m}(x,y)-f(x,y))]_{y=\eta_0}\\
&\quad+\mathrm{D}_y^m[Q_{n-1,m}(x,y)Q_{n-1,m-1}(x,y)(f(x,y)-T_{n-1,m-1}(x,y))]_{y=\eta_0}\}.
\end{aligned} \tag{4.8}$$

Thus, from (4.5) and (4.8)

$$\begin{aligned}
&T_{n-1,m}(x,y)-T_{n-1,m-1}(x,y)=\\
&=\frac{\omega_m^2(y)}{m!Q_{n-1,m}(x,y)Q_{n-1,m-1}(x,y)}\\
&\quad\times\mathrm{D}_y^m\{Q_{n-1,m}(x,y)Q_{n-1,m-1}(x,y)[f(x,y)-T_{n-1,m-1}(x,y)]\}_{y=\eta_0}\\
&\quad-\frac{\omega_n^1(x)\omega_m^2(y)}{n!m!Q_{n-1,m}(x,y)Q_{n-1,m-1}(x,y)}\\
&\quad\times\mathrm{D}_{x,y}^{n+m}[Q_{n-1,m}(x,y)Q_{n-1,m-1}(x,y)f(x,y)]_{y=\eta_0;x=\xi_0}.
\end{aligned} \tag{4.9}$$

If

$$\Phi(x,y)=Q_{n-1,m}(x,y)Q_{n-1,m-1}(x,y)[f(x,y)-T_{n-1,m-1}(x,y)]$$

we obtain the following theorem.

**Theorem 2.** *If for the function $f:\Delta\to\mathbb{R}$, $f\in\mathcal{C}_\Delta^{n+m}$, all partial inverted differences are defined, then*

$$f(x,y)-T_{n-1,m-1}(x,y)=\frac{1}{Q_{n-1,m}(x,y)Q_{n-1,m-1}(x,y)}$$

$$\times \left\{ \frac{\omega_n^1(x)}{n!} \mathrm{D}_x^n [\Phi(\xi_0, y)] + \frac{\omega_m^2(y)}{m!} \mathrm{D}_y^m [\Phi(x, \eta_0)] - \frac{\omega_n^1(x)}{n!} \frac{\omega_m^2(y)}{m!} \mathrm{D}_{x,y}^{n+m} [\Phi(\xi_0, \eta_0)] \right\}. \tag{4.10}$$

**Remark.** If we put $y = y_k$, $1 \leqslant k \leqslant m$ into (4.10), then

$$f(x, y_k) - T_{n-1,m-1}(x, y_k) = \frac{1}{[Q_{n-1,m-1}(x, y)]^2} \frac{\omega_n^1(x)}{n!} \times \mathrm{D}_x^n \{ [Q_{n-1,m-1}(x, y_k)]^2 [f(x, y_k) - T_{n-1,m-1}(x, y_k)] \}_{x=\xi_0} \tag{4.11}$$

and then (4.11) is the well-known error formula for interpolating the restricted function $f(\cdot, y_k)$ by a rational one-variable function with degrees of polynomials in the numerator and the denominator not greater respectively than $[\frac{1}{2}n]$ and $[\frac{1}{2}(n-1)]$.

## 5. Algorithm for finding coefficients of interpolating BCF

As in Section 3, let $\Pi_x^n \times \Pi_y^m = \{x_0, x_1, \ldots, x_n\} \times \{y_0, y_1, \ldots, y_m\}$ be a rectangular grid and let $f(x, y)$ be a function defined in the rectangle $\Delta$. If there exists a BCF interpolating the function $f(x, y)$ on our grid then it has the form

$$T_{n,m}(x, y) = \left[ b_0^0 + \frac{y - y_0}{b_1^0} + \cdots + \frac{y - y_{m-1}}{b_m^0} \right] + \frac{x - x_0}{\left[ b_0^1 + \frac{y - y_0}{b_1^1} + \cdots + \frac{y - y_{m-1}}{b_m^1} \right]} + \cdots + \frac{x - x_{n-1}}{\left[ b_0^n + \frac{y - y_0}{b_1^n} + \cdots + \frac{y - y_{m-1}}{b_m^n} \right]}. \tag{5.1}$$

The problem of finding $T_{n,m}(x, y)$ such that

$$T_{n,m}(x_i, y_j) = f(x_i, y_j), \quad i = 0, 1, \ldots, n, j = 0, 1, \ldots, m, \tag{5.2}$$

is reduced to finding $(n+1)(m+1)$ coefficients $b_j^i$ of BCF (5.1).

We assume that for a given integer $k$, $0 \leqslant k \leqslant n$,

$$t_m^k(y) \equiv b_m^k \tag{5.3}$$

and for $l = m-1, \ldots, 1, 0$,

$$t_l^k(y) = b_l^k + \frac{y - y_l}{t_{l+1}^k(y)}. \tag{5.4}$$

Similarly

$$T_n(x, y) \equiv t_0^n(y), \tag{5.5}$$

and for $k = n-1, \ldots, 1, 0$,

$$T_k(x, y) = t_0^k(y) + \frac{x - x_k}{T_{k+1}(x, y)}. \tag{5.6}$$

Thus

$$T_k(x_k, y) = t_0^k(y), \quad k = 0, 1, \ldots, n-1 \tag{5.7}$$

and

$$t_l^k(y_l) = b_l^k, \quad l = 0, 1, \ldots, m-1. \tag{5.8}$$

Now, from (5.8), (5.3) and (5.4)

$$t_{l+1}^k(y) = \frac{y - y_l}{t_l^k(y) - t_l^k(y_l)}, \quad l = 0, 1, \ldots, m-1, \tag{5.9}$$

and from (3.6)

$$T_{k+1}(x, y) = \frac{x - x_k}{T_k(x, y) - t_0^k(y)}, \quad k = 0, 1, \ldots, n-1. \tag{5.10}$$

Therefore, we will divide our algorithm for finding $b_l^k$ into two main stages. In the first one we will find values $t_0^k(y_j)$, $j = 0, 1, \ldots, m$, for $k = 0, 1, \ldots, n$. In the second one we will find values $t_j^k(y_j) = b_j^k$, $j = 0, 1, \ldots, m$, for each $0 \leqslant k \leqslant n$.

From (5.4) and (5.6) we have

$$T_0(x_0, y_j) = t_0^0(y_j) = f_{0j}, \quad j = 0, 1, \ldots, m, \tag{5.11}$$

and for $i = 0, 1, \ldots, n$

$$T_0(x_i, y_j) = f_{ij}, \quad j = 0, 1, \ldots, m. \tag{5.12}$$

Thus

$$T_1(x_1, y_j) = t_0^1(y_j) = \frac{x_1 - x_0}{T_0(x_1, y_j) - t_0^0(y_j)}, \quad j = 0, 1, \ldots, m. \tag{5.13}$$

Now, assuming that we have computed values $t_0^k(y_j)$ for $k = 1, 2, \ldots, p-1 < n$, we obtain $t_0^p(y_j)$ from the sequence of equations

$$\begin{aligned}
T_1(x_p, y_j) &= \frac{x_p - x_0}{T_0(x_p, y_j) - t_0^0(y_j)}, \\
T_2(x_p, y_j) &= \frac{x_p - x_1}{T_1(x_p, y_j) - t_0^1(y_j)}, \\
&\vdots \\
T_{p-1}(x_p, y_j) &= \frac{x_p - x_{p-2}}{T_{p-2}(x_p, y_j) - t_0^{p-2}(y_j)}, \\
T_p(x_p, y_j) = t_0^p(y_j) &= \frac{x_p - x_{p-1}}{T_{p-1}(x_p, y_j) - t_0^{p-1}(y_j)}
\end{aligned} \tag{5.14}$$

where $j = 0, 1, \ldots, m$.

If values $t_0^p(y_0), t_0^p(y_1), \ldots, t_0^p(y_m)$ are computed we will find the coefficients $b_j^p$, $j = 0, 1, \ldots, m$, using

Table 1

| | $x_i = 5°$ | $x_i = 15°$ | $x_i = 25°$ | $x_i = 35°$ |
|---|---|---|---|---|
| $y_j = 30°$ | 0.5238 | 0.5251 | 0.5277 | 0.5313 |
| $y_j = 35°$ | 0.6111 | 0.6133 | 0.6173 | 0.6231 |
| $y_j = 40°$ | 0.6985 | 0.7016 | 0.7077 | 0.7162 |
| $y_j = 45°$ | 0.7859 | 0.7903 | 0.7987 | 0.8109 |

Table 2

| | $j=0$ | $j=1$ | $j=2$ | $j=3$ |
|---|---|---|---|---|
| $t_0^0(y_j)$ | 0.5238 | 0.6111 | 0.6985 | 0.7859 |
| $t_0^1(y_j)$ | 7692.3076 | 4545.4545 | 3225.8064 | 2272.7272 |
| $t_0^2(y_j)$ | −0.0039 | −0.0076 | −0.0095 | −0.0141 |
| $t_0^3(y_j)$ | −6666.66667 | −4545.4545 | −2814.5229 | −2191.0604 |

Table 3

| | $y=32°$ | $v=34°$ | $y=36°$ | $y=38°$ | $y=40°$ | $y=42°$ |
|---|---|---|---|---|---|---|
| $x=10°$ | 0.0002 | 0.0000 | 0.0002 | 0.0004 | 0.0004 | 0.0001 |
| $x=20°$ | 0.0003 | 0.0010 | 0.0025 | 0.0019 | 0.0021 | 0.0015 |
| $x=30°$ | 0.0000 | 0.0023 | 0.0024 | 0.0046 | 0.0050 | 0.0052 |

the Thacher–Tukey algorithm [12] that is, we will build a triangle table

$$\begin{array}{cccc} \boldsymbol{t_0^k(y_0)}, & t_0^k(y_1) & ,\ldots, & t_0^k(y_m) \\ & \boldsymbol{t_1^k(y_1)} & ,\ldots, & t_1^k(y_m) \\ & & \ddots & \vdots \\ & & & \boldsymbol{t_m^k(y_m)} \end{array} \tag{5.15}$$

according to the rule

$$t_{l+1}^k(y) = \frac{y - y_l}{t_l^k(y) - t_l^k(y_l)}, \tag{5.16}$$

$0 \leqslant k \leqslant n$, $l = 0, 1, \ldots, m-1$. The bold values $t_j^k(y_j)$ are from (5.8) the coefficients of the continued fraction, located on the $k$th stage of BCF (5.1).

Obviously, this algorithm is working if all the terms in (5.14) and in the table (5.15) are finite. It is easy to modify the algorithm (by permutations of points in $\Pi_x^n$ and $\Pi_y^m$) in order to check whether the interpolating BCF exists.

**Example 1.** For the elliptic integral

$$F(x, y) = \int_0^y (1 - \sin^2 x \sin^2 t)^{-1/2} \mathrm{d}t \tag{5.17}$$

we have the values at respective points given in Table 1.

Then the values $t_0^p(y_j)$, $p, j = 0, 1, 2, 3$ are given in Table 2 and the interpolating BCF has the form

$$\begin{aligned} T_{3,3}(x, y) = &\left[0.5238 + \frac{y-30}{57.2738} + \frac{y-35}{-152.4855} + \frac{y-40}{-0.0131}\right] \\ &+ \frac{x-5}{\left[7692.3076 + \dfrac{y-30}{-0.0048} + \dfrac{y-35}{3846.1538} + \dfrac{y-40}{-0.0019}\right]} \\ &+ \frac{x-15}{\left[-0.0039 + \dfrac{y-30}{-4078.3033} + \dfrac{y-35}{0.0044} + \dfrac{y-40}{-1666.6667}\right]} \\ &+ \frac{x-25}{\left[-6666.6667 + \dfrac{y-30}{0.0071} + \dfrac{y-35}{-2222.2222} + \dfrac{y-40}{0.0036}\right]}. \end{aligned}$$

We have computed the table of errors $E(x, y)=|T_{3,3}(x, y)-F(x, y)|$ at some points. These values are given in Table 3.

The relative error $E(x, y)/F(x, y)$ at these points is not greater than 0.007.

## 6. Approximation with branched continued fractions. Partial reciprocal derivatives

Let $\rho_{k,l}(x_0, x_1,\dots, x_k; y_0, y_1,\dots, y_l)$ be the $(k, l)$th partial reciprocal difference for the function $f(x, y)$ on the rectangular grid $\Pi_x\times\Pi_y\subset\Delta$. If the limit

$$\lim_{y_0,y_1,\dots,y_l\to\bar{y}}\left[\lim_{x_0,x_1,\dots,x_k\to\bar{x}}\rho_{k,l}(x_0,\dots,x_k;y_0,\dots,y_l)\right]=R_y^lR_x^kf(\bar{x},\bar{y}) \tag{6.1}$$

exists and is finite then it is called the $(k, l)$th partial reciprocal derivative for $f(x, y)$ at the point $(\bar{x},\bar{y})\in\Delta$.

In particular

$$R_y^0R_x^0f(x,y)=f(x,y), \tag{6.2}$$

$$R_y^0R_x^1f(x,y)=R_x^1f(x,y)=\frac{1}{\mathrm{D}_xf(x,y)},\qquad R_y^1R_x^0f(x,y)=R_y^1f(x,y)=\frac{1}{\mathrm{D}_yf(x,y)}. \tag{6.3}$$

If the function $f(x, y)$ has its $(k, l)$th partial reciprocal derivatives for $k=0, 1,\dots, n$, $k=0, 1,\dots, m$ at the point $(\bar{x},\bar{y})\in\Delta$ then from Lemma 2 there exists a limit

$$\varphi_{k,l}(\bar{x},\bar{y})=\lim_{y_0,y_1,\dots,y_l\to\bar{y}}\left[\lim_{x_0,x_1,\dots,x_k\to\bar{x}}\varphi_{k,l}(x_0,\dots,x_k;y_0,\dots,y_l)\right]$$

and for $k\geqslant 2$

$$\varphi_{k,0}(\bar{x},\bar{y})=R_x^kf(\bar{x},\bar{y})-R_x^{k-2}f(\bar{x},\bar{y}). \tag{6.4}$$

Using the symmetry of partial reciprocal differences we have

$$\begin{aligned}\varphi_{k,0}(\bar{x},y)&=\lim_{x_0,\dots,x_k\to\bar{x}}\frac{x_k-x_{k-1}}{\varphi_{k-1,0}(x_0,\dots,x_{k-2},x_k;y)-\varphi_{k-1,0}(x_0,\dots,x_{k-2},x_{k-1};y)}\\&=\frac{1}{\mathrm{D}_{x_{k-1}}\left[\rho_{k-1,0}(x_0,x_1,\dots,x_{k-1};y)\right]_{x_0=\dots=x_{k-1}=\bar{x}}}.\end{aligned} \tag{6.5}$$

On the other hand

$$\begin{aligned}\mathrm{D}_x\left[R_x^kf(x,y)\right]&=\sum_{i=0}^{k}\mathrm{D}_{x_i}\left[\rho_{k,0}(x_0,\dots,x_{i-1},x_{i+1},\dots,x_k,x_i;y)\right]_{x_0=\dots=x_k=\bar{x}}\\&=(k+1)\mathrm{D}_{x_k}\left[\rho_{k,0}(x_0,x_1,\dots,x_k;y)\right]_{x_0=\dots=x_k=\bar{x}}\end{aligned} \tag{6.6}$$

Therefore, for $k\geqslant 2$

$$\varphi_{k,0}(x,y)=\frac{k}{\mathrm{D}_x\left[R_x^{k-1}f(x,y)\right]}. \tag{6.7}$$

Similarly, for $l\geqslant 2$

$$\begin{aligned}\varphi_{k,l}(\bar{x},\bar{y})&=\lim_{\substack{y_i\to\bar{y}\\ i=0,\dots,l}}\left[\lim_{\substack{x_j\to\bar{x}\\ j=0,\dots,k}}\varphi_{k,l}(x_0,\dots,x_k;y_0,\dots,y_l)\right]\\&=\lim_{\substack{y_i\to\bar{y}\\ i=0,\dots,l}}\left[\lim_{\substack{x_j\to\bar{x}\\ j=0,\dots,k}}\frac{y_l-y_{l-1}}{\rho_{k,l-1}(x_0,\dots,x_k;y_0,\dots,y_{l-2},y_l)-\rho_{k,l-1}(x_0,\dots,x_k;y_0,\dots,y_{l-2},y_{l-1})}\right]\end{aligned}$$

$$= \frac{1}{\mathrm{D}_{y_{l-1}}\left[\rho_{k,l-1}(x_0,\ldots,x_k;y_0,\ldots,y_{l-1})\right]_{y_0=\cdots=y_{l-1}=\bar{y},x_0=\cdots=x_k=\bar{x}}}$$

$$= \frac{l}{\mathrm{D}_y\left[R_y^{l-1}R_x^k f(x,y)\right]}. \tag{6.8}$$

From (6.7) and (6.8) we can easily compute values $\varphi_{n,m}(x,y)$ because if we put $R_x^{-1}f(x,y) = R_y^{-1}R_x^k f(x,y) = 0$ then

$$R_x^k f(x,y) = R_x^{k-2} f(x,y) + \varphi_{k,0}(x,y), \tag{6.9}$$

$$\varphi_{k+1}(x,y) = \frac{k+1}{\mathrm{D}_x\left[R_x^k f(x,y)\right]} \tag{6.10}$$

and

$$R_y^l R_y^k f(x,y) = R_y^{l-2} R_x^k f(x,y) + \varphi_{k,l}(x,y), \tag{6.11}$$

$$\varphi_{k,l+1}(x,y) = \frac{l+1}{\mathrm{D}_y\left[R_y^l R_x^k f(x,y)\right]}. \tag{6.12}$$

Let for the function $f(x, y)$ all the partial reciprocal derivatives $R_y^l R_x^k f(x,y)$ exist for $k, l = 0, 1,\ldots$ in the rectangle $\Delta$. If $T_{n,m}(x, y)$ is a rational function of the form (4.2) and if it interpolates $f(x,y)$ on the rectangular grid $\Pi_x^n \times \Pi_y^m$ then putting $y_0,\ldots,y_m \to \bar{y}$, $x_0,\ldots,x_n \to \bar{x}$, we obtain a rational function of the form

$$\begin{aligned} R_{n,m}(x,y) = {} & \left[\varphi_{0,0}(\bar{x},\bar{y}) + \frac{y-\bar{y}}{\varphi_{0,1}(\bar{x},\bar{y})} + \cdots + \frac{y-\bar{y}}{\varphi_{0,m}(\bar{x},\bar{y})}\right] \\ & + \frac{x-\bar{x}}{\left[\varphi_{1,0}(\bar{x},\bar{y}) + \frac{y-\bar{y}}{\varphi_{1,1}(\bar{x},\bar{y})} + \cdots + \frac{y-\bar{y}}{\varphi_{1,m}(\bar{x},\bar{y})}\right]} \\ & + \cdots + \frac{x-\bar{x}}{\left[\varphi_{n,0}(\bar{x},\bar{y}) + \frac{y-\bar{y}}{\varphi_{n,1}(\bar{x},\bar{y})} + \cdots + \frac{y-\bar{y}}{\varphi_{n,m}(\bar{x},\bar{y})}\right]}. \end{aligned} \tag{6.13}$$

**Theorem 3.** *Let the function $f: \Delta \to \mathbb{R}$ be analytic in the neighbourhood of the point $(x_0, y_0) \in \Delta$, let there exist partial reciprocal derivatives $R_y^l R_x^k f(x, y)$, $k = 0, 1,\ldots, n$, $l = 0, 1,\ldots, m$, and $\varphi_{k,l}(x_0, y_0) \neq 0$, $k = 0, 1,\ldots, n$, $l = 0, 1,\ldots, m$. Then in the Taylor series for the difference*

$$f(x,y) - R_{n,m}(x,y) = \sum_{k,l=0}^{\infty} d_{k,l}(x-x_0)^k(y-y_0)^l \tag{6.14}$$

*where $R_{n,m}(x, y)$ is the rational function of the form (6.13) with $\bar{x} = x_0$, $\bar{y} = y_0$ all the coefficients $d_{k,l}$ with $k = 0, 1,\ldots, n$, $l = 0, 1,\ldots, m$ are equal to zero.*

**Proof.** Since $\varphi_{k,l}(x_0, y_0) \neq 0$, we have $Q_{n,m}(x_0, y_0) \neq 0$ where $Q_{n,m}(x, y)$ is a polynomial in the denominator of $R_{n,m}(x, y)$. Now the proof follows from Theorem 2. □

If some of $\varphi_{k,l}(x_0, y_0)$ vanish or are infinite we can obtain a similar result for the Taylor series for $[f(x, y)Q_{n,m}(x, y) - P_{n,m}(x, y)]$. Such cases will be called irregular.

**Example 2.** For

$$f(x,y) = (1+x+y)^{1/2} \tag{6.15}$$

we have

$$\varphi_{0,0}(x, y) = (1 + x + y)^{1/2}, \qquad \varphi_{k,0}(x, y) = \varphi_{0,k}(x, y) = 2(1 + x + y)^{1/2},$$
$$\varphi_{k,2j-1}(x, y) = (1 + x + y)^{1/2}, \qquad \varphi_{k,2j}(x, y) = 4(1 + x + y)^{1/2},$$

$k, j = 1, 2, \ldots$. Therefore if $x_0 = y_0 = 0$ then the BCF for (6.15) has the form

$$(1 + x + y)^{1/2} = \left[1 + \frac{y}{2} + \frac{y}{2} + \ldots\right] + \frac{x}{\left[2 + \frac{y}{1} + \frac{y}{4} + \frac{y}{1} + \frac{y}{4} + \ldots\right]} + \frac{x}{\left[2 + \frac{y}{1} + \frac{y}{4} + \frac{y}{1} + \frac{y}{4} + \ldots\right]} + \cdots. \tag{6.16}$$

**Example 3.** For

$$f(x, y) = \log(1 + x + y) \tag{6.17}$$

we have

$$\varphi_{0,0}(x, y) = \log(1 + x + y), \qquad \varphi_{0,2k}(x, y) = \frac{2}{k}, \qquad \varphi_{0,2k-1}(x, y) = (2k - 1)(1 + x + y),$$
$$\varphi_{2k,0}(x, y) = \frac{2}{k}, \qquad \varphi_{2k,1}(x, y) = \infty, \qquad \varphi_{2k-1,0}(x, y) = (2k - 1)(1 + x + y),$$
$$\varphi_{2k-1,1}(x, y) = \frac{1}{2k - 1}, \qquad \varphi_{2k-1,2}(x, y) = \infty,$$

$k = 1, 2, \ldots$. Therefore the BCF corresponding to (6.17) has the form

$$\begin{aligned}\log(1 + x + y) = &\left[\frac{y}{1} + \frac{y}{2} + \frac{y}{3} + \ldots + \frac{y}{2k - 1} + \frac{y}{2/k} + \ldots\right] \\ &+ \frac{x}{2} + \frac{x}{1 + y} + \ldots + \\ &+ \frac{x}{2/k} + \frac{x}{2k - 1 + y/\{1/(2k - 1)\}} + \cdots.\end{aligned} \tag{6.18}$$

The BCF corresponding to $\log(1 + x + y)$ is very specific, since all branches except the first one are finite. It is an example of an irregular BCF.

From Theorem 3 rational functions obtained as approximants of BCF corresponding the given function can be treated as Padé-type approximants for two-variable functions. In particular, we can use it for

Table 4

| $n$ | $C_{n,n}(2, 2)$ | $K_{2n+1}(2.2)$ | $T_{2n+1}(2, 2)$ | $C_{n,n}(5, 5)$ | $K_{2n+1}(5, 5)$ | $T_{2n+1}(5, 5)$ |
|---|---|---|---|---|---|---|
| 2 | 0.4455 | 0.45 | 0.4454632 | 0.041 | 0.312 | 0.2996907 |
| 3 | 0.4483 | 0.452 | 0.4476771 | 0.17 | 0.326 | 0.3011913 |
| 4 | 0.44718 | 0.44706 | 0.4472448 | −0.27 | 0.3001 | 0.3014559 |
| 5 | 0.44720 | 0.447293 | 0.4472159 | 8.49 | 0.3035 | 0.3015013 |
| 6 | 0.447215 | 0.4472156 | 0.4472137 | 0.287 | 0.30153 | 0.3015095 |
| 8 | | | | 0.065 | 0.301695 | 0.3015112 |
| | $f = 0.4472136$ | | | $f = 0.3015113$ | | |

Table 5

| $n$ | $C_{n,n}(1, 2)$ | $K_{2n+1}(1, 2)$ | $T_{2n+1}(1, 2)$ | $C_{n,n}(5, 5)$ | $K_{2n+1}(5, 5)$ | $T_{2n+1}(5, 5)$ |
|---|---|---|---|---|---|---|
| 2 | 0.053 | 0.067 | 0.040449 | 0.011 | 0.042 | 0.0097008 |
| 3 | 0.049713 | 0.04984 | 0.0498002 | 0.000035 | −0.0022 | 0.009015 |
| 4 | 0.0497882 | 0.04982 | 0.0497868 | 0.000060 | 0.0029 | 0.000185 |
| 5 | 0.0497871 | 0.0497872 | 0.0497870 | 0.0000446 | 0.00032 | 0.0000462 |
| 6 | | 0.0497871 | 0.0497871 | 0.0000454 | 0.00037 | 0.0000461 |
| | $f = 0.0497871$ | | | $f = 0.0000454$ | | |

finding approximate values of functions. Since other types of two-variable Padé-type approximants can be used for this purpose, we will give a comparison of results obtained with Canterbury approximants and $S$-continued fractions defined by O'Donohoe and Kutschminskaja.

Let $C_{n,n}(x, y)$ denote the $[n, n]$th Canterbury approximant and $K_n(x, y)$ the $n$-th approximant of O'Donohoe continued fraction for the given function $f(x, y)$.

**Example 4.** If $g(x, y) = (1 + x + y)^{1/2}$, then the BCF corresponding to the function has the form (6.16). Let $T_{n,m}(x, y)$ be the $(n, m)$th approximant of BCF (6.16), and let $T_k(x, y) = 1/T_{k,k}(x, y)$. Table 4 illustrates the speed of convergence of corresponding Padé-type approximants to the exact value of the function $f(x, y) = (1 + x + y)^{-1/2}$, which is denoted by $f$ and which is given with seven decimal places precision. Here we examine odd approximants $T_{2n+1}$ and $K_{2n+1}$ since the number of coefficients corresponds to the number of coefficients of the Canterbury diagonal approximant $C_{n,n}$. The values of $K_{2n+1}$ and $C_{n,n}$ are taken from a paper by Murphy and O'Donohoe [9].

**Example 5.** For function $f(x, y) = \exp[-(x + y)]$ we have the values of Table 5.

**Example 6.** The BCF corresponding to the function $f(x, y) = \log(1 + x + y)$ can be used for accelerating the speed of convergence of values of the continued fraction for $\log(1 + t)$ at points which are far from the origin. If we put $x = 0$ in (6.18) then this BCF becomes an ordinary continued fraction expansion for the function $\log(1 + y)$. Therefore we will substitute the first branch in (6.18) by the value of $\log(1 + y_0)$ computed with the given precision and than we will use approximants of the BCF corresponding to $\log(1 + x + y)$ for finding approximate value of $\log(1 + x_0 + y_0)$. For example if we put $x_0 = 5$, $y_0 = 5$, then for $m = 12$ we have the values of respective approximants of BCF (6.18) shown in Table 6.

In the right part of Table 6 we give values of the $n$-th approximant of the continued fraction corresponding to $\log(1 + t)$ with $t_0 = 10$ (the $(n/n)$th Padé approximant for $f(t) = \log(1 + t)$). The exact value of log 11 equals 2.397895.... We observe quite fast convergence of $T_{n,m}$ approximants while Padé approximants are converging slowly in this case. It is connected with the fact that the distance from the point (5, 5) to the origin (on the plane) is smaller than the distance from $t_0 = 10$ to the origin (on the line).

Table 6

| $n$ | $T_{n,12}(5, 5)$ | $n$ | $\Pi_{n,n}(10)$ |
|---|---|---|---|
| 2 | 2.3799 | 1 | 1.7 |
| 4 | 2.3974 | 2 | 2.2 |
| 6 | 2.39774 | 3 | 2.32 |
| 8 | 2.39782 | 4 | 2.377 |
| 10 | 2.3978943 | 5 | 2.3918 |
| 12 | 2.3978958 | 6 | 2.3961 |

## 7. Final remarks

Examples given in Sections 5 and 6 show that branched continued fractions can be used for solving two-dimensional interpolation and approximation problems. On the other hand, we have now a wide choice of two-variable rational Padé-type approximants which can be used for the same purpose. Which one is the best?

The answer is complicated because it is difficult to find simple connections between all types of such approximants. In our case, we have easy algorithms for finding coefficients of BCF solving the given interpolation problem of BCF corresponding to the given elementary function.

Another advantage of BCFs is that if we restrict the function $f(x, y)$ to the line parallel to the $x$-axis, then the restriction of a respective approximant of the BCF corresponding to $f(x, y)$, i.e. $T_{n,m}(x, y)|_{y=y_0}$ is the $([\frac{1}{2}(n+1)]/[\frac{1}{2}n])$th Padé approximant for the one-variable function $f(\cdot, y_0)$. It gives a possibility of obtaining convergence theorems generalising known results for one-variable Padé approximants.

Since partial inverted differences can simply be generalized for functions of $N \geqslant 2$ variables we will easily find respective connections and theorems for $N$-variable functions. Moreover, all the above-presented constructions can be used for functions of two or more complex variables. Obviously, the error formula 4.10 is not valid in this case, but it is easy to prove Theorem 3 for complex analytic functions and therefore approximants of respective BCF can be treated as Padé-type approximants for a given complex-valued function.

## References

[1] J.S.R. Chisholm and J. McEvan, *Proc. Roy. Soc. London A* **336** (1974) 421–452.

[2] A.A. Goncar, *Mat.Sb.T.* **93** (135) (2) (1974) 296–313 (in Russian).

[3] R. Hughes Jones, *J. Approx. Theory* **16** (3) (1976) 201–235.

[4] C.H. Lutterodt, *J. Math. Anal. Appl.* **53** (1976) 89–98.

[5] J. Karlsson and H. Wallin, in: E.B. Saff and R.S. Varga, Eds., *Padé and Rational Approximations* (Academic Press, London, 1977) 83–100.

[6] A.M. Cuyt, in: M.G. de Bruin and H. van Rossum, Eds., *Padé Approximation and its Application,* (Springer, Berlin, 1981) 127–157.

[7] P.I. Bodnarcuk and W.Ja. Skorobogatko, *Branched Continued Fractions and Applications* (Naukowaja Dumka, Kiev, 1974) (in Ukrainian).

[8] K.J. Kutschminskaja, *Dokl. Akad. Nauk USSR Ser. A* **7** (1978) 614–617.

[9] J.A. Murphy and M.R. O'Donohoe, *J. Comput Appl. Math.* **4** (3) (1978) 181–190.

[10] W. Siemaszko, *J. Comput. Appl. Math.* **6** (2) (1980) 121–125.

[11] F.B. Hildebrand, *Introduction to Numerical Analysis* (McGraw-Hill, New York, 1974).

[12] G.A.Jr. Baker and P. Graves-Morris, *Encyclopedia of Mathematics and its Applications, Padé Approximants,* Vol. I, II (Addison-Wesley, London, 1981).